DETERMINATION OF THE CONSTANTS IN EULER'S PROBLEM CONCERNING THE MINIMUM AREA BETWEEN A CURVE AND ITS EVOLUTE.*

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The evolute problem in the calculus of variations is stated as follows: Given two points P_0 and P_1 and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area.

The problem was first given by Euler in his Methodus Inveniendi, etc., p. 64, 1744. It has also been discussed by several later writers, among whom may be mentioned Lindelöf and Moigno† and Kneser.‡

The only curves which can give a minimum are known to be cycloids. The question then naturally arises: *Is it possible to join any two points a and b by a cycloid which is tangent to two given directed lines through a and b?* This is the question which will here be discussed.

For convenience the equation of the cycloid will be taken in the form

$$C: \begin{cases} x = t - \sin t = \pi + 2u + \sin 2u, \\ y = -1 + \cos t = -(1 + \cos 2u), \end{cases}$$

where u is the angle through which the tangent line has turned, starting from $-\pi/2$ (see figure 1). The first arch will then correspond to the interval

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2}.$$

The following problem is then proposed: Is it possible to find two points A and B on this cycloid such that the tangents at these points intersect at the same angle as the two given directed lines through a and b? If this problem can be solved, then the triangle ABC can be transformed by a similarity transformation into a triangle of the same size as abc and then by a movement of the plane it can be made to coincide with abc itself. Both of these transformations transform cycloids into cycloids and the original cycloid C will therefore go into one which satisfies the conditions of the original problem.

In order to find the angles α and β for which there exists a triangle ABC on the cycloid C, one may derive the values of the angles α, β in terms of the parameter values u, v of the points A and B, and then study the transformation of the uv-plane into the $\alpha\beta$-plane.   Different cases arise according as the arc AB contains a cusp or not.   In § 1 the arc AB is supposed to con--
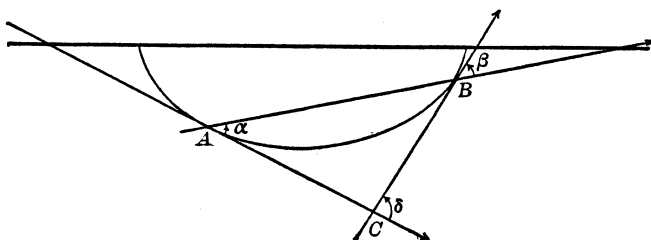

FIG. 1.

tain no cusp, and for convenience it will be taken on the first arch of the cycloid.   This is the only case admitted by the variation problem for it has been shown that if the arc contains a cusp it does not furnish a minimum. In § 2 AB is supposed to contain n cusps and the discussion is divided into two cases according as δ is positive or negative.

§ 1.   Solution of the problem when the arc AB contains no cusps.

Let u and v be the parameter values of A and B and therefore in this case the angles which the positive tangents at A and B make with the positive x-axis, φ the angle which the line AB makes with the x-axis and α, β, δ the angles indicated in Fig. 1.

If u and v are the parameter values of $A(x_1, y_1)$ and $B(x_2, y_2)$ respectively then we have

(1) $$\tan \varphi = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sin (v - u) \sin (v + u)}{v - u + \sin (v - u) \cos (v + u)}.$$

From the figure it readily follows that the transformation of the uv-plane into the $\alpha\beta$-plane has the form

(2) $$\begin{cases} \alpha = \varphi - u, \\ \beta = v - \varphi. \end{cases}$$

The uv region which it is proposed to transform is T_1 of Fig. 2, defined by the inequalities

(3) $$-\frac{\pi}{2} \leqq u < v, \quad -\frac{\pi}{2} < v \leqq \frac{\pi}{2},$$

and corresponding to points A and B on the first arch of the cycloid.

Consider the line MN

(4) $v - u = \delta,$

where δ is a constant.   It is transformed into the line $M'N'$

(5) $\alpha + \beta = \delta,$
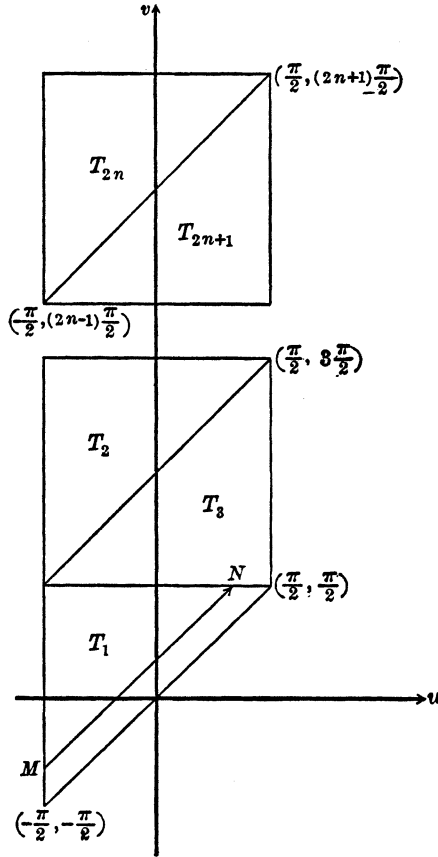

FIG. 2.

(see Fig. 3).   The values of α corresponding to points on this line are defined by the equation

(6) $\tan \alpha = \tan (\varphi - u) = \dfrac{- \delta \sin u + \sin \delta \sin (u + \delta)}{\delta \cos u + \sin \delta \cos (u + \delta)} \equiv F(u).$

On the line (4) u has the limits

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2} - \delta.$$

For these values $F(u)$ decreases monotonically from a positive maximum value α_1 to a positive minimum value α_2. For of the three expressions

$$(7) \qquad F\left(-\frac{\pi}{2}\right) = \frac{\delta - \sin \delta \cos \delta}{\sin^2 \delta} = \tan \alpha_1,$$

$$(8) \qquad F\left(\frac{\pi}{2} - \delta\right) = \frac{\sin \delta - \delta \cos \delta}{\delta \sin \delta} = \tan \alpha_2,$$

$$F'(u) = \frac{\sin^2 \delta - \delta^2}{[\delta \cos u + \sin \delta \cos (u + \delta)]^2},$$

the first two are positive and the third negative for δ between its limiting values 0 and π (see Fig. 1).

It is readily seen that the numerators are positive or negative by differentiating each until a derivative of constant sign is found, and considering the initial values. For example suppose we consider the numerator of $F'(u)$,

$$N(\delta) = \sin^2 \delta - \delta^2, \quad N'(\delta) = 2(\sin \delta \cos \delta - \delta),$$
$$N''(\delta) = -2 \sin^2 \delta.$$

The second derivative is always negative or zero and so it follows that the function N itself has a maximum equal to zero for $\delta = 0$. The quotient $F(u)$ cannot become infinite for $0 < \delta < \pi$, $-\pi/2 \gtreqless u \gtreqless \pi/2 - \delta$, for both u and $u + \delta$ are then between $-\pi/2$ and $+\pi/2$. Consequently both terms in the denominator of $F'(u)$ are always positive and they cannot vanish simultaneously.

By a comparison of the equations (7) and (8) it is easily verified that

$$(9) \qquad \tan \alpha_2 = \tan (\delta - \alpha_1).$$

This shows that the figure is symmetric with respect to the bisector of the first quadrant of the $\alpha\beta$-plane. So having computed α_1 the range of values of α follows. Then from the equation (5) one finds the end values of β. By means of the formulæ (7), (8), (9) the table given below is readily computed.

Table A.

δ	$\alpha_1 = \beta_2$	$\alpha_2 = \beta_1$
0.	0.	0.
.524	.348	.176
.755	.472	.283
1.047	.686	.361
1.571	1.004	.567
2.094	1.280	.814
2.356	1.397	.959
2.618	1.489	1.129
3.142	1.571	1.571

In this table the angles are expressed in radians. From these values the image I in the $\alpha\beta$-plane of the triangle T_1 in the uv-plane can be plotted. It is found to have the form shown in Fig. 3.*
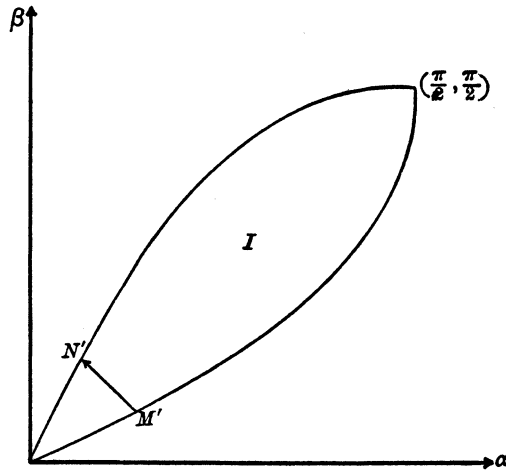

FIG. 3.

As the point (u, v) monotonically describes parallel lines MN of the triangle T_1 in the uv-plane, the corresponding point (α, β) describes monotonically the parallel lines $M'N'$ shown in the region I. It is evident that the correspondence between T_1 and its image I in the $\alpha\beta$-plane is a one-to-one correspondence. One has then the following theorem: *If the two angles α and β are given, for which the point (α, β) lies in the region I, then there exists one, and only one, pair of points A and B on the same arch of the cycloid C for which the triangle ABC has the given angles. If the point (α, β) lies outside of I there is no pair of points A and B which will give the required angles.*

The connection is now readily made with the original problem. *When the two points a and b and the directed lines ac and cb through them are given, there is either one or no cycloid joining a and b, having the tangents ac and cb and containing no cusp on the arc ab. If the angles α and β are given δ is found by means of relation (5), and then α_1 and α_2 can be determined by the equations (7), (8), (9). If α lies between α_1 and α_2 the construction can be made, otherwise not.*

§ 2.　Cycloid arcs containing n cusps.

When the extension is made to arcs containing n cusps it will be found convenient to suppose that the point A is on the first arch of the cycloid

* The point $(0, 0)$ of the $\alpha\beta$-plane is the only point of image I to be excluded, the remainder of the boundary being included. This point corresponds to the value $\delta = 0$ and would require that the two points A and B coincide.

to the right of the y-axis. The values of u and v corresponding to the points A and B will always lie in the intervals

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2}, \quad (2n-1)\frac{\pi}{2} \leqq v \leqq (2n+1)\frac{\pi}{2}.$$

If then α, β, δ are defined as the angles obtained in going from AC to AB, AB to CB and AC to CB through the angle whose absolute value is less than or equal to π, the following expressions for α and β in terms of u, v and φ are obtained

(10)
$$\begin{cases} \alpha = \varphi - u, \\ \beta = v - \varphi - n\pi. \end{cases}$$

Hence the range of values of the angle δ defined by the equation

(11)
$$\delta = \alpha + \beta = v - u - n\pi$$

is readily verified to be from $-\pi$ to $+\pi$. Likewise the equation which defines α is

(12)
$$\tan \alpha = \frac{\sin \delta \sin (u+\delta) - (n\pi + \delta) \sin u}{(n\pi + \delta) \cos u + \sin \delta \cos (u+\delta)},$$

this being analogous to equation (6) of §1.

Denoting the right-hand side of equation (12) by $G(u)$ it is at once seen that its derivative $G'(u)$,

$$G'(u) = \frac{\sin^2 \delta - (n\pi + \delta)^2}{[(n\pi + \delta) \cos u + \sin \delta \cos (u+\delta)]^2},$$

is always negative or zero.

For further consideration it will be necessary to make a division of cases according as δ is positive or negative.

Case I. $0 \leqq \delta \leqq \pi$.

In case δ is positive all the admissible values of the point (u, v) lie in the triangle T_{2n} (Fig. 2). From this it follows that for a given constant value of δ between 0 and π the values of u for points on the line $v = u + \delta + n\pi$ in the triangle T_{2n} have the range

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2} - \delta.$$

In this interval the derivative $G'(u)$ is negative and therefore the function decreasing. Moreover the denominator of $G(u)$ is always positive, since

each of its terms is positive in the given interval. The numerator, on the
other hand, vanishes at most once, as is seen from the formula
$$\sin u \sin (u + \delta) - (n\pi + \delta) \sin u$$

$$= (n\pi + \delta - \sin \delta \cos \delta) \cos u \left\{ \frac{\sin^2 \delta}{n\pi + \delta - \sin \delta \cos \delta} - \tan u \right\}.$$

As for the limiting values of $G(u)$ it is easily seen that the expression

$$G\left(-\frac{\pi}{2}\right) = \frac{n\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta}$$

is always positive for all values of δ in the range $0 \gtrless \delta \gtrless \pi$ and becomes
infinite when $\delta = 0$, while the expression

$$G\left(\frac{\pi}{2} - \delta\right) = \frac{-(n\pi + \delta) \cos \delta + \sin \delta}{(n\pi + \delta) \sin \delta}$$

is sometimes positive and sometimes negative, according to the value of δ.
It has the value $-\infty$ when $\delta = 0$ and the value $+\infty$ for $\delta = \pi$. Hence,
*for a given positive value of δ the angle α decreases continuously from a value
$\alpha_1(0 < \alpha_1 \gtrless \pi/2)$ to a value $\alpha_2(-\pi/2 \gtrless \alpha_2 \gtrless \pi/2)$ Then from the relation
$\alpha + \beta = \delta$ it follows that there is but one value of β corresponding to each of
these values of α.*

The image of the line $v = u + \delta + n\pi$ in the $\alpha\beta$-plane is the segment of
the line $\alpha + \beta = \delta$ between the ordinates α_1 and α_2. The locus of the points
(α_1, β_1), (α_2, β_2) for the different values of δ can be plotted by means of
the formulae

(13) $$\tan \alpha_1 = \frac{n\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta},$$

(14) $$\tan \alpha_2 = \frac{\sin \delta - (n\pi + \delta) \cos \delta}{(n\pi + \delta) \sin \delta},$$

which satisfy the relation

(15) $$\tan \alpha_2 = \tan (\delta - \alpha_1) = \tan \beta_1.$$

This last formula again indicates symmetry with respect to the bisectors
of the first and third quadrants of the plane.

From these last equations it is seen that for any fixed n the curves traced
by the points (α_1, β_1), (α_2, β_2) lie in the interior of the isosceles triangle whose
vertices are $(-\pi/2, \pi/2)$, $(\pi/2, \pi/2)$, $(\pi/2, -\pi/2)$ and have these three
points in common. Moreover, if a comparison of results is made when two
particular values of n are substituted in equations (13) and (14) it is seen

that the curves flatten towards the lines $\alpha = \pi/2$, $\beta = \pi/2$ when n increases. In other words the curve defined by these equations for a definite value of n includes the curve for the same equations when n is replaced by $n - 1$. In particular will this be the case for $n = 0$, as is readily verified by comparing equations (13) and (14) with (7) and (8).

Hence, *when the angles α and β of the triangle ABC define a point (α, β) in the region defined by the equations (13), (14), (15), there is one and but one pair of points A and B on the first and $(n + 1)$th arches of the cycloid for which ABC has the required angles. Therefore, if any, there is but one cycloid arc containing n cusps and passing through the points a and b in the required directions. If the point (α, β) does not lie in this region there is no solution of the original problem.*

In order to see if a construction is possible when α and β, and therefore δ, are given, it is only necessary to find the values of α_1 amd α_2 from the formulæ (13) and (14) and see if the given value of α lies between these two.

Finally it should be noticed that when δ is positive, any two points a and b which can be joined by an arc containing no cusps can always be joined by an arc containing n cusps, but not conversely. This conclusion follows at once from the preceeding statements regarding the regions in the $\alpha\beta$-plane.
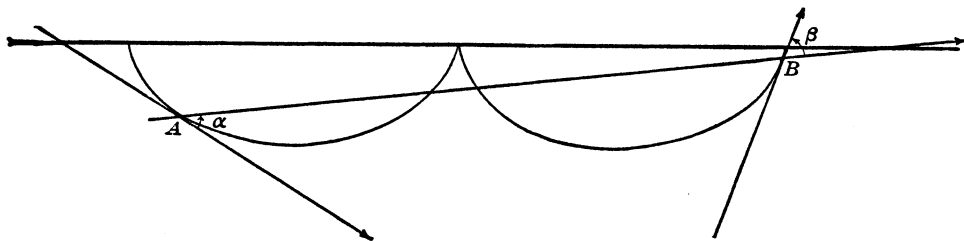

FIG. 4.

As a special example illustrating these results consider the case when $n = 1$. Then, as follows from Fig. 4, it is seen that u and v lie in the intervals

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2}, \quad \frac{\pi}{2} \leqq v \leqq \frac{3\pi}{2}.$$

Further the relations (10) and (11) now become

(10′)
$$\begin{cases} \alpha = \varphi - u, \\ \beta = v - \varphi - \pi, \end{cases}$$

and

(11′)
$$\delta = \alpha + \beta = v - u - \pi,$$

respectively, from which it follows that u actually has the range

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2} - \delta.$$

Finally the formulæ for computing (α_1, β_1) and (α_2, β_2) derived from (13) and (14) for $n = 1$ are

(13') $$\tan \alpha_1 = \frac{\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta},$$

(14') $$\tan \alpha_2 = \frac{\sin \delta - (\pi + \delta) \cos \delta}{(\pi + \delta) \sin \delta}.$$

From these relations table B can be computed.

Table B.

δ	$\alpha_1 = \beta_2$	$\beta_1 = \alpha_2$
0.0	1.571	−1.571
.524	1.493	− .969
.785	1.427	− .642
1.047	1.374	− .327
1.228	1.356	− .134
1.309	1.329	− .020
1.396	1.353	.043
1.571	1.361	.210
2.094	1.414	.680
2.356	1.486	.870
2.618	1.530	1.088
3.142	1.571	1.571

From this table it is found that the image II in the $\alpha\beta$-plane of the triangle T_2 in the uv-plane has the form shown in Fig. 5.

Case II. $-\pi \leqq \delta \leqq 0$.

In case the arc AB is to contain n cusps but δ is negative the admissible region of the uv-plane is the triangle T_{2n+1} (Fig. 2). From the graph of the line $v = u + \delta + n\pi$ it follows that the admissible values of u for a constant value of δ between $-\pi$ and 0 are

$$-\frac{\pi}{2} - \delta \leqq u \leqq \frac{\pi}{2}.$$

In this interval it is known that the derivative $G'(u)$ is always negative and consequently $G(u)$, and therefore α, decreases continuously as u increases. In order to find the admissible values of α we will first discuss its end values when $u = -\pi/2 - \delta$ and $u = \pi/2$.

Consider first the expression for the end value $G(-\pi/2 - \delta)$,

$$G\left(-\frac{\pi}{2} - \delta\right) = \frac{(n\pi + \delta) \cos \delta - \sin \delta}{-(n\pi + \delta) \sin \delta}.$$

Its numerator has the positive derivative $- (n\pi + \delta) \sin \delta$, and assumes the value $- (n - 1)\pi$ when $\delta = - \pi$. For further discussion it is now necessary to distinguish two cases according as $n > 1$ or $n = 1$. If $n > 1$ the numerator varies from $- (n - 1)\pi$ to $n\pi$ as δ varies from $- \pi$ to 0. But the denominator is always positive, being zero at the end values $-\pi$ and 0. Hence it follows that the expression $G(- \pi/2 - \delta)$ varies from $- \infty$ to $+ \infty$ and becomes zero but once for δ between $- \pi$ and 0. If $n = 1$ it is at once seen that $G(- \pi/2 - \delta)$ varies only between 0 and $+ \infty$ for this range of values.

Consider next the expression $G(\pi/2)$,

$$G\left(\frac{\pi}{2}\right) = \frac{(n\pi + \delta) - \sin \delta \cos \delta}{\sin^2 \delta}.$$

The numerator of this fraction has the positive derivative $1 - \cos 2\delta$ and varies from $(n - 1)\pi$ to $n\pi$ as δ varies from $- \pi$ to 0. So both numerator and denominator of $G(\pi/2)$ are positive, therefore $G(\pi/2)$ itself, and it has the value $+ \infty$ for δ equal both $- \pi$ and 0 if $n > 1$. If $n = 1$ the end values of $G(\pi/2)$ are 0 and $+ \infty$.

Furthermore $G(u)$ can vanish but once in the interval $-\pi/2 - \delta \gtreqless u \gtreqless \pi/2$, if at all. In order to see this it is only necessary to examine the zeros of the numerator which we will denote by $N(u)$. The derivative of this

$$N'(u) = \sin \delta \cos (u + \delta) - (n\pi + \delta) \cos u$$

is always negative. In fact each term of the derivative is always negative as is readily verified when it is remembered that $u + \delta = v - n\pi$.

If $n = 1$ this numerator $N(u)$ has the positive value

$$(\pi + \delta) \cos \delta - \sin \delta$$

when $u = -\pi/2 - \delta$ and the negative value

$$- (\pi + \delta) + \sin \delta \cos \delta$$

when $u = \pi/2$. Hence there is one and only one value of u making $G(u)$ vanish. From this it follows that there is only one value of u making $G(u)$ infinite. For, as has been shown, $G(u)$ is a decreasing function of u such that both end values are positive, and there is one point of vanishing. Hence $G(u)$ becomes negative and must therefore become infinite if its end value is positive.

On the other hand, if $n > 1$, it is found that, if the numerator is considered as a function of δ when u is replaced by $- \pi/2 - \delta$, it then has a positive derivative and changes continuously from a negative to a positive

value as δ varies from $-\pi$ to 0. Thus it happens that, for some values of δ, the function $G(u)$ does not become zero but starts from a negative value and decreases through infinity to a positive value $G(\pi/2)$.

Hence the following statements can be made: *If $n = 1$ the angle α decreases for a given δ from a value $\alpha_1(0 \leqq \alpha_1 \leqq \pi/2)$ through zero and $-\pi/2$ to a value $\alpha_2 (-\pi \leqq \alpha_2 \leqq -\pi/2)$. To each of these values there corresponds*
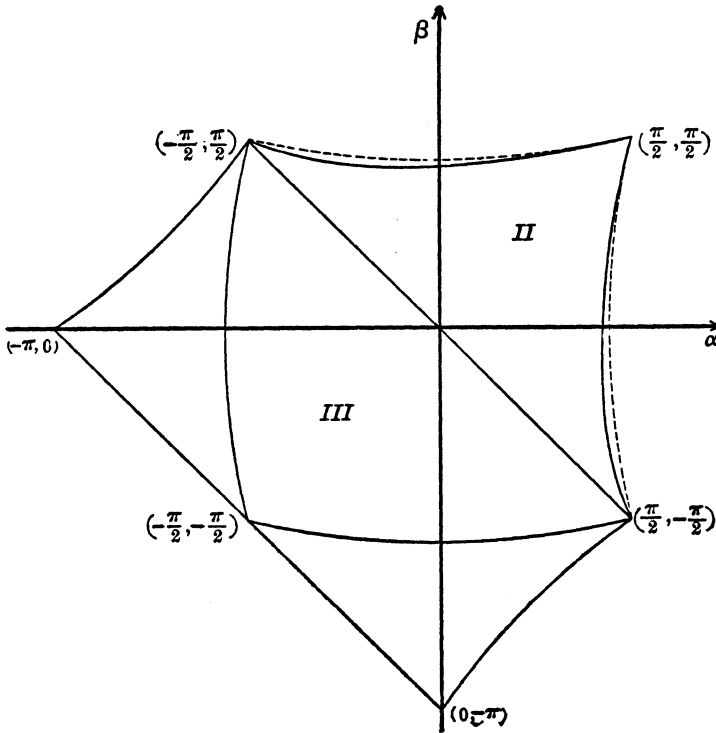

FIG. 5.

but one value of β. If $n > 1$ the angle α decreases from a value $\alpha_1(-\pi/2 \leqq \alpha_1 \leqq \pi/2)$ to a value α_2 less than $-\pi/2$.

The points (α_1, β_1), (α_2, β_2) defining the admissible region of the $\alpha\beta$-plane are then given by the formulæ

$$\text{(13'')} \qquad \tan \alpha_1 = \frac{\sin \delta - (\pi + \delta) \cos \delta}{(\pi + \delta) \sin \delta},$$

$$\text{(14'')} \qquad \tan \alpha_2 = \frac{\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta},$$

$$\text{(15'')} \qquad \tan \alpha_2 = \tan (\delta - \alpha_1),$$

in case $n = 1$; and by the formulæ

$$(13''') \qquad \tan \alpha_1 = \frac{\sin \delta - (n\pi + \delta) \cos \delta}{(n\pi + \delta) \sin \delta},$$

$$(14''') \qquad \tan \alpha_2 = \frac{n\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta},$$

$$(15''') \qquad \tan \alpha_2 = \tan (\delta - \alpha_1)$$

in case $n > 1$.

From the formulæ $(13'')$, $(14'')$, $(15'')$ the table C can be computed and a continuous graph joining the points $(0, -\pi)$, $(\pi/2, -\pi/2)$ and $(-\pi, 0)$, $(-\pi/2; \pi/2)$ is found* as shown in III, Fig. 5.

Table C.

δ	$\alpha_1 = \beta_2$	$\alpha_2 = \beta_1$
0.000	1.571	−1.571
−0.524	1.129	−1.653
−0.785	.960	−1.745
−1.047	.812	−1.859
−1.571	.567	−2.138
−2.094	.429	−2.523
−2.356	.268	−2.624
−2.618	.176	−2.794
−3.142	.000	−3.142

As an illustration of the case $n > 1$ let n have the particular value $n = 2$. The formulæ $(13''')$, $(14''')$, $(15''')$ then allow us to compute the tables D from which follows the continuous graph joining the points $(-\pi/2, -\pi/2)$, $(-\pi/2, \pi/2)$ and $(-\pi/2, -\pi/2)$, $(\pi/2, -\pi/2)$ as shown in III, Fig. 5.

Table D.

δ	$\alpha_1 = \beta_2$	$\alpha_2 = \beta_1$
−0.000	1.571	−1.571
−0.524	1.083	−1.607
−0.785	0.869	−1.654
−1.047	0.653	−1.700
−1.571	0.208	−1.779
−2.094	−0.333	−1.761
−2.356	−0.639	−1.717
−2.618	−0.970	−1.648
−3.142	−1.571	−1.571

By a comparison of the preceding formulæ it is seen that, if $n > 1$, then, for an increasing number of cusps, the curves defining the boundary

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* In case $n = 1$ and $\delta = -\pi$ the points A and B must coincide at the cusp, as is readily verified geometrically. Hence this value of δ must be excluded.

region of the $\alpha\beta$-plane flatten towards the lines

$$\beta = -\frac{\pi}{2}, \quad \alpha = -\frac{\pi}{2}.$$

We therefore conclude that *when the angles α and β of the triangle ABC define for a negative δ a point (α, β) in the region* III *corresponding to the given number of cusps, n, then there is one, and but one, pair of values (u, v) in the region T_{2n+1} (Fig. 2) and therefore one and but one pair of points A and B in the first and $(n + 1)$th arches of the cycloid for which the triangle has the required angles. Hence there is one and but one cycloid arc containing n cusps and passing through the points a and b in the directions prescribed by the original problem. If for a given n the point (α, β) lies outside its region in* III *no solution of the original problem exists.*

Finally it follows from the conclusions regarding the regions in the $\alpha\beta$-plane that for a negative δ, *any two points a and b which can be joined by an arc containing n cusps can always be joined by one containing $n - 1$ cusps, but not conversely.*

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